

The University of Chicago

ON THE WARING PROBLEM WITH
POLYNOMIAL SUMMANDS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1935

By
MABEL GWENETH HUMPHREYS

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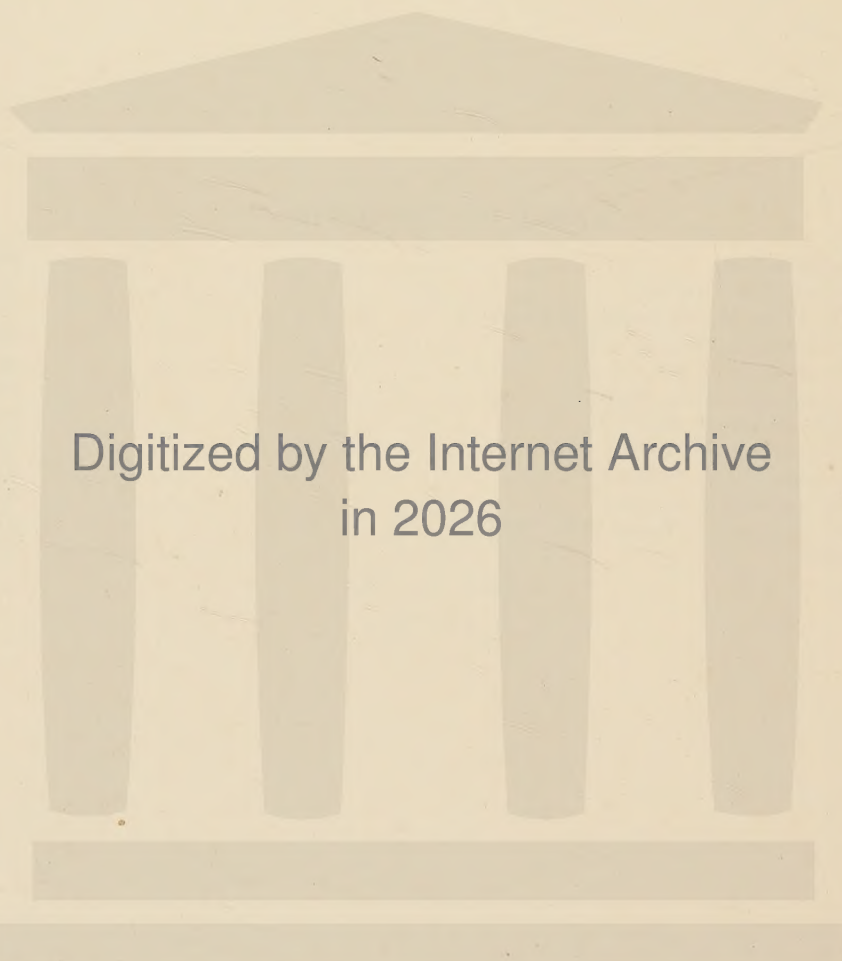
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ON THE WARING PROBLEM WITH POLYNOMIAL SUMMANDS

BY M. GWENETH HUMPHREYS

1. Introduction. When Waring, in 1770, made his famous conjecture concerning the representation of all positive integers as sums of powers of positive integers, he also suggested the possibility of a similar representation by polynomials. Very little was done toward examining this until recently. L. E. Dickson¹ has proved by algebraic methods that every integer greater than a stated integer is expressible as a sum of nine values of a cubic polynomial in which the square term does not appear. Supplementing this proof by table work he has proved universal theorems² for special cubic polynomials. Universal theorems have also been proved for special cubic polynomials by Frances E. Baker³ and G. C. Webber.⁴

E. Landau⁵ has proved by analytic methods similar to those of Hardy and Littlewood a theorem for polynomials of degree k which corresponds to their first theorem for k -th powers.⁶ R. D. James⁷ has completed the analysis and proved that every sufficiently large integer (greater than a finite integer not determined) is a sum of nine values of a polynomial of the form

$$a(x^3 - x)/6 + b(x^2 - x)/2 + cx,$$

where $a > 0$, $(a, b, c) = 1$, and $a \not\equiv 4c \pmod{8}$.

In the present paper, with the analysis as developed by Landau and James, it is proved that, after excluding exceptional cases similar to $a \not\equiv 4c \pmod{8}$ above, every sufficiently large integer can be represented by $s \geq (k-2)2^{k-1} + 5$ values of a polynomial of degree k when $k = 4, 5, 6$, or 7 . If the polynomial satisfies a further condition, this theorem is proved for $k \leq 28$.

2. Introductory analysis. Let

$$\phi(x) = \beta_k x^k + \beta_{k-1} x^{k-1} + \cdots + \beta_1 x + \beta_0$$

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¹ L. E. Dickson, Transactions of the American Mathematical Society, vol. 36 (1934), pp. 731-748.

² L. E. Dickson, Transactions of the American Mathematical Society, vol. 36 (1934), pp. 1-12.

³ Frances E. Baker, Dissertation, Chicago, 1934.

⁴ G. C. Webber, Transactions of the American Mathematical Society, vol. 36 (1934), pp. 493-510.

⁵ E. Landau, *Über die neue Winogradoffsche Behandlung des Waring'schen Problems*, Mathematische Zeitschrift, vol. 31 (1930), pp. 319-338.

⁶ E. Landau, *Vorlesungen über Zahlentheorie*, Leipzig, 1927.

⁷ R. D. James, American Journal of Mathematics, vol. 56 (1934), pp. 303-315.

be a polynomial of degree k with integral coefficients such that $\beta_k > 0$. Let $r_s(n)$ denote the number of integral solutions of

$$n = \sum_{\nu=1}^s \phi(x_\nu), \quad x_\nu \geq 0.$$

Then E. Landau⁵ has proved that

$$(1) \quad \left| r_s(n) - \frac{\Gamma^s(k + 1/k)}{\beta_k^{s/k} \Gamma(s/k)} \mathfrak{S} n^{(s-k)/k} \right| < C_1 n^{(s-k-\delta)/k}$$

for $s \geq (k-2)2^{k-1} + 5$, where C_1 and δ are positive constants depending only on $s, \beta_k, \dots, \beta_0$. The function $\mathfrak{S}$ is called the singular series and is defined as follows. Let r be prime to q ,

$$\rho = e^{2\pi i r/q}, \quad S_\rho = \sum_{h=0}^{q-1} \rho^{\phi(h)}, \quad A(q) = \sum_{\substack{r=0 \\ (r,q)=1}}^{q-1} q^{-s} S_\rho^s e^{-2\pi i r n/q}.$$

Then

$$\mathfrak{S} = \sum_{q=1}^{\infty} A(q).$$

If $\mathfrak{S} \geq \eta > 0$, where η is independent of n , then from (1), $r_s(n) > 0$ when

$$(2) \quad n > C_2 = \left[\frac{C_1 \beta_k^{s/k} \Gamma(s/k)}{\eta \Gamma^s(k + 1/k)} \right]^{k/\delta}.$$

Let $\theta = \theta(p)$ be the highest power of p which divides every coefficient of $\phi'(x)$, and $\gamma = \gamma(p)$ be defined by

$$\gamma = \begin{cases} \theta + 2, & p = 2, \\ \theta + 1, & p > 2. \end{cases}$$

Denote $p^{-\theta}\phi'(x)$ by $\phi_0(x)$. Let $M(m) = M(m, n)$ be the number of integral solutions of

$$(3) \quad \sum_{\nu=1}^s \phi(x_\nu) \equiv n \pmod{m}, \quad 0 \leq x_\nu < m.$$

For $m = p^l$, let $N(p^l) = N(p^l, n)$ be the number of integral solutions of this congruence in which at least one $\phi_0(x_\nu)$ is prime to p . Such solutions will be called primitive. Then the following lemmas hold.⁷

LEMMA 1. If $l \geq \gamma$, then

$$N(p^l) = p^{(l-\gamma)(s-1)} N(p^\gamma).$$

LEMMA 2. $M(m) = m^{s-1} \sum_{q|m} A(q)$.

LEMMA 3. If p_h denotes the h -th prime, then

$$\sum_{q/p_1^l \dots p_l^l} A(q) = \prod_{p \leq p_l} \sum_{q/p^l} A(q).$$

3. **The form of polynomials of degree k .** A polynomial⁸ $F(y)$ of degree k which represents an integer for every integral value of $y \geq 0$ represents an integer for every integral value of y . Then set

$$(4) \quad F(y) = a_k \begin{bmatrix} y \\ k \end{bmatrix} + a_{k-1} \begin{bmatrix} y \\ k-1 \end{bmatrix} + \cdots + a_1 \begin{bmatrix} y \\ 1 \end{bmatrix} + a_0,$$

where $a_k, \dots, a_0$ are rational, and $\begin{bmatrix} y \\ s \end{bmatrix} = y(y+1) \cdots (y+s-1)/s!$.

Since $F(y)$ is an integer for all integral values of y , it is for $y = 0$. Therefore a_0 is an integer. Similarly it is seen by setting $y = -1, \dots, -k+1$, that $a_1, \dots, a_k$ are all integers.

Conversely, if $a_k, \dots, a_0$ are all integers, $F(y)$ is an integer for all integral values of y .

Therefore every polynomial of degree k in y which represents an integer for all integral values of $y \geq 0$ can be expressed in form (4) where $a_k, \dots, a_0$ are integers.

For the following discussion, $a_k > 0$, and a_0 may be taken equal to zero. For if $n = \sum_{\nu=1}^s F(y_\nu)$ has an integral solution for $n > C_2$, then $n = \sum_{\nu=1}^s P(y_\nu)$ has an integral solution for $n > C_2 - sa_0$, where $P(y) = F(y) - a_0$. It is also assumed that $a_k, \dots, a_1$ are such that for no prime p is $P(y) \equiv 0 \pmod{p}$ for all y , for if this were so for some p , all sums of values of $P(y)$ would be multiples of p . This last condition will be referred to as hypothesis I. It implies $(a_k, \dots, a_1) = 1$.

It will be useful to write also

$$P(y) = (b_k y^k + \cdots + b_1 y)/k!,$$

where

$$(5) \quad \begin{aligned} b_{k-r} &= \sum_{q=0}^r d_{qr} a_{k-q}, \\ d_{00} &= 1, \quad d_{rr} = k(k-1) \cdots (k-r+1), \quad d_{0r} = \sum_{n_1, \dots, n_r=1}^{k-1} n_1 \cdots n_r \\ &\quad (n_i \neq n_j; \quad i, j = 1, \dots, r), \\ d_{qr} &= k(k-1) \cdots (k-q+1) \sum_{n_1, \dots, n_{r-q}=1}^{k-q-1} n_1 \cdots n_{r-q} \\ &\quad (n_i \neq n_j; \quad i, j = 1, \dots, r-q; q = 1, \dots, r-1). \end{aligned}$$

In $P(y)$ set $y = vx + t$ to give

$$Q(x) = A_k x^k + \cdots + A_1 x + A_0,$$

⁸ David Hilbert, *Mathematische Annalen*, vol. 36 (1890), pp. 511, 512.

where

$$(6) \quad \begin{aligned} A_k &= b_k v^k / k!, \\ A_{k-r} &= \left\{ \sum_{l=0}^{r-1} \frac{(k-l) \cdots (k-r+1)}{(r-l)!} b_{k-l} t^{r-l} + b_{k-r} \right\} v^{k-r} / k!, \end{aligned}$$

$t \geq 0$ is a parameter which is assigned integral values depending on $a_k, \dots, a_1$, and $v \geq 0$ is an integer having the smallest number of factors p such that $A_k, \dots, A_0$ are integers for every value of t . For $p > k, p \nmid v$.

By hypothesis I, there are values of t such that there is at least one value of x for which $p \nmid Q(x)$. Such values of t will be called *admissible*. If $p \nmid v$, all values of t are admissible.

Since $Q(x)$ has integral coefficients, $\theta(p)$ as defined in §2 can be determined for each p .

4. Congruence (3) for $p > k$. As in §2, $r_s(n) > 0$ for $n > C_2$, if $\mathfrak{S} \geq \eta > 0$. To prove that for $s \geq (k-2)2^{k-1} + 5$, $\mathfrak{S} \geq \eta > 0$, we prove that for this value of s and for all primes p and integers $n, N(p^\gamma) > 0$. This proof is divided into two parts according as $p > k$ or $p \leq k$.

For primes $p > k$ we prove

LEMMA 4. *If $3 < k \leq 28, s \geq (k-2)2^{k-1} + 5$, and $p > k$, then $N(p^\gamma) \geq 1$.*

For $p > k, p \nmid v$. By hypothesis I, at least one coefficient of $Q(x)$ is not divisible by p . Then $\theta = 0, \gamma = 1$, and the congruence (3) becomes

$$\sum_{v=1}^s Q(x_v) \equiv n \pmod{p}.$$

L. E. Dickson⁹ has proved that for $k \leq 28, p > k$, and $\sigma = s - 1$, the congruence

$$\sum_{v=1}^{\sigma} Q(x_v) \equiv n \pmod{p}$$

has at least one integral solution.

Not all the coefficients of $Q_0(x)$ are divisible by p , and therefore there is at least one value of x for which $p \nmid Q_0(x)$. Suppose $p \nmid Q_0(x_1)$. Then from the above

$$\sum_{v=1}^{\sigma} Q(x_v) \equiv n - Q(x_1) \pmod{p}$$

has an integral solution. Thus

$$\sum_{v=1}^s Q(x_v) \equiv n \pmod{p}$$

⁹ L. E. Dickson, American Journal of Mathematics, July, 1935.

has a primitive solution, that is,

$$N(p^\gamma) = N(p) \geq 1 \quad \text{for} \quad s \geq (k-2)2^{k-1} + 5,$$

for every n and every $p > k$.

5. Congruence (3) for $p \leq k$. When $p \leq k$, $Q_0(x)$ is congruent mod p to a polynomial of degree $p-1$ obtained from $Q_0(x)$ by replacing x^p by x . If for every admissible value of t , all the coefficients of the polynomial so obtained are divisible by p , there obviously can be no primitive solutions of (3). We shall say that $P(y)$ satisfies hypothesis II if for each prime $p \leq k$ there is an admissible value of t such that for some value of x , $p \nmid Q_0(x)$. Hypothesis II may be stated also in terms of the coefficients $A_k, \dots, A_1$: for each prime $p \leq k$ there is an admissible value of t for which the congruences

$$\begin{aligned} A_1 &\equiv 2A_2 + (p+1)A_{p+1} + \dots \\ &\equiv \dots \equiv pA_p + (2p-1)A_{2p-1} + \dots \\ &\equiv 0 \end{aligned} \pmod{p^{\theta+1}}$$

are not all satisfied.

The theorems of the paper will be proved for polynomials which satisfy hypotheses I and II. For $k=4$ and 5 conditions on the coefficients a of $P(y)$ which determine whether hypothesis II is satisfied are stated. For polynomials of higher degrees such conditions are not stated.

For the discussion of congruence (3) in this case we prove two lemmas.

LEMMA 5. *If $p \nmid Q(x_1)$, $p \nmid Q_0(x_2)$, then $N(p^\gamma) \geq 1$ for $s \geq p^\gamma + 1$ and every integer n .*

Any integer is congruent mod p^γ to a sum of at most p^γ values $Q(x_1)$. If $x_2 \equiv x_1 \pmod{p}$, the lemma follows. If $x_2 \not\equiv x_1 \pmod{p}$, $n - Q(x_2)$ is congruent mod p^γ to a sum of at most p^γ values $Q(x_1)$, and the lemma follows.

LEMMA 6. *If $p \nmid (A_k, \dots, A_1)$, $p \nmid Q(x_1)$, $p \nmid Q_0(x_2)$, then $N(p^\gamma) \geq 1$ for $s \geq (k-2)2^{k-1} + 5$ and every integer n .*

Let $p^{w-1} \leq k < p^w$. Then since $p \nmid (A_k, \dots, A_1)$, $\theta \leq w-1$. By Lemma 5, $N(p^\gamma) \geq 1$ for $s \geq p^\gamma + 1$.

For $p > 2$,

$$p^\gamma + 1 \leq p^w + 1 \leq pk + 1 \leq k^2 + 1 < (k-2)2^{k-1} + 5.$$

Therefore $N(p^\gamma) \geq 1$ for $s \geq (k-2)2^{k-1} + 5$.

For $p = 2$,

$$p^\gamma + 1 \leq p^{w+1} + 1 \leq 4k + 1 \leq (k-2)2^{k-1} + 5.$$

Therefore $N(p^\gamma) \geq 1$ for $s \geq (k-2)2^{k-1} + 5$.

6. Completion of the analysis. By means of the Lemmas 4, 5, and 6 it will be proved that for polynomials of degree 4, 5, 6, and 7 satisfying hypoth-

eses I and II, and for certain polynomials of degree k , $7 < k \leq 28$, satisfying the same hypotheses, that $N(p^\gamma) \geq 1$ for $s \geq (k-2)2^{k-1} + 5$ and every integer n .

From this fact and Lemmas 1 and 2, since $M(p^l) \geq N(p^l)$ we have

$$\begin{aligned} \sum_{q/p^l} A(q) &= p^{-l(s-1)} M(p^l) \geq p^{-l(s-1)} N(p^l) \\ (7) \qquad &= p^{-l(s-1)} p^{(l-\gamma)(s-1)} N(p^\gamma) \geq p^{-\gamma(s-1)} \\ &\geq p^{-\gamma_1(s-1)}, \end{aligned}$$

where γ_1 is for each value of k the greatest value of γ such that $2^\gamma \leq (k-2)2^{k-1} + 4$.

Also, by Landau,⁵ Theorem 8, with $\epsilon = 1/8$

$$\begin{aligned} \sum_{q/p^l} A(q) &= 1 + \sum_{\lambda=1}^l A(p^\lambda) > 1 - E \sum_{\lambda=1}^\infty p^{-9\lambda/8} \\ (8) \qquad &= 1 - E(p^{9/8} - 1)^{-1}. \end{aligned}$$

Finally, it has been proved by Landau⁵ that $\mathfrak{S}$ is absolutely convergent for $s \geq (k-2)2^{k-1} + 5$. Hence, by (7), (8), and Lemma 3

$$\begin{aligned} \mathfrak{S} &= \lim_{l \rightarrow \infty} \sum_{q/p^l} A(q) = \lim_{l \rightarrow \infty} \prod_{p \leq p^l} \sum_{q/p^l} A(q) \\ &\geq \prod_p \max(p^{-\gamma_1(s-1)}, 1 - E(p^{9/8} - 1)^{-1}) \\ &\geq \eta > 0. \end{aligned}$$

As indicated in §1, it follows that $r_s(n) > 0$ for $n > C_2$.

7. Polynomials of degree 4. We prove

THEOREM 1. *If $P(y)$ satisfies hypotheses I and II, every sufficiently large integer can be represented as a sum of 21 values of a polynomial*

$$P(y) = a_4 \left[\frac{y}{4} \right] + a_3 \left[\frac{y}{3} \right] + a_2 \left[\frac{y}{2} \right] + a_1 \left[\frac{y}{1} \right],$$

$a_4, \dots, a_1$ being integers, $a_4 > 0$. $P(y)$ fails to satisfy hypothesis II when and only when one or more of the following three sets of simultaneous conditions hold:

- 1) $3|(a_4, a_3)$, $3^2|a_4$, $2a_3 + 3a_2 + 6a_1 \equiv 2a_4 + 3a_2 \equiv 0 \pmod{9}$,
- 2) $2^3|a_4$, $2^4|a_4$, $2|(a_3, a_2)$,
 $a_4 + a_3 + 6a_2 \equiv a_4 + 2a_3 \equiv 0 \pmod{2^4}$,
 $3a_4 + 4a_3 + 6a_2 + 12a_1 \equiv 0 \pmod{2^5}$,
- 3) $2^4|a_4$, $2|(a_3, a_2)$, $a_2 + 2a_1 \equiv a_3 + 2a_2 \equiv 0 \pmod{8}$.

First we prove that for $P(y)$ as in the theorem, and $s \geq 21$, $N(p^r) \geq 1$.

For $p > k$, this is proved in Lemma 4.

Let $p \leq k$. Formulas (5) and (6) give for $k = 4$,

$$b_4 = a_4, \quad b_3 = 6a_4 + 4a_3, \quad b_2 = 11a_4 + 12a_3 + 12a_2,$$

$$b_1 = 6a_4 + 8a_3 + 12a_2 + 24a_1,$$

$$A_4 = b_4 v^4 / 4!, \quad A_3 = (4b_4 t + b_3) v^3 / 4!, \quad A_2 = (6b_4 t^2 + 3b_3 t + b_2) v^2 / 4!,$$

$$A_1 = (4b_4 t^3 + 3b_3 t^2 + 2b_2 t + b_1) v / 4!, \quad A_0 = P(t).$$

A. Case $p = 3$.

1. Suppose $3/(a_4, a_3)$, then $3/v$. By hypothesis I, every value of t is admissible.
 - a. Unless $b_1 \equiv 4b_4 + 2b_2 \equiv 0 \pmod{9}$, there is a value of t for which $3/A_1$. Then $\theta = 0$ and $3/Q_0(0)$. Then by Lemma 5, $N(3) \geq 1$ for $s \geq 4$.
 - b. If $b_1 \equiv 4b_4 + 2b_2 \equiv 0 \pmod{9}$, for every value of t , $A_1 \equiv 4A_4 + 2A_2 \equiv 0 \pmod{3}$. If $9/b_4$, $\theta = 0$ and the polynomial does not satisfy hypothesis II. This case gives exception 1 of the theorem. If $9/b_4$, then $9/b_3$ and $3/A_3$. Then $\theta = 1$ and there is a value of x for which $3/Q_0(x)$. By Lemma 5, $N(3^2) \geq 1$ for $s \geq 10$.
2. Suppose $3/(a_4, a_3)$, then $v \equiv 3, 6 \pmod{9}$. $3/(A_4, A_3, A_2)$ and there is a value of t such that $3/A_1$. Then $\theta = 0$, $3/Q_0(0)$, and either $3/Q(0)$ or $3/Q(1)$ according as $3/A_0$ or $3/A_0$. By Lemma 5, $N(3) \geq 1$ for $s \geq 4$.

B. Case $p = 2$.

1. Suppose $2^3/a_4$, $2/(a_3, a_2)$, then $2/v$. By hypothesis I, every value of t is admissible.

Unless $b_1 \equiv b_3 \equiv 0 \pmod{16}$, there is a value of t for which $2/A_1$. Then $\theta = 0$ and $2/Q_0(0)$. By Lemma 5, $N(2^2) \geq 1$ for $s \geq 5$.

If $b_1 \equiv b_3 \equiv 0 \pmod{16}$, then $2/A_1$ for all t . There are two cases to be considered.

 - a. If $16/a_4$, then $2/A_4$, $2/(A_3, A_2, A_1)$. Also $\theta \leq 2$ and $2/Q(1)$ or $2/Q(0)$ according as $2/A_0$ or $2/A_0$. Then by Lemma 5, $N(2^r) \geq 1$ for $s \geq 17$, unless $\theta = 2$ and for every value of t , $A_1 \equiv 4A_4 + 3A_3 + 2A_2 \equiv 0 \pmod{8}$, which gives exception 2 of the theorem.
 - b. If $16/a_4$, then $2/A_4$, $2/A_2$, and $\theta = 1$. Then $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. If for some t , $4/A_1$, then $2/Q_0(0)$. Then $N(2^3) \geq 1$ for $s \geq 9$. If $4/A_1$ for all values of t , there is no value of t for which $4/A_3$ and the polynomial is under exception 3 of the theorem.
2. Suppose $2/a_4$ but not all the conditions of Case B1 hold. Then $v \equiv 2 \pmod{4}$ and $2/(A_4, A_3)$.
 - a. Suppose $2a_3 \equiv a_4 + 2a_2 \equiv 0 \pmod{4}$ does not hold.

If $4/a_4$, then $2/A_2$ and there is a value of t for which $2/A_1$. Then

$\theta = 0$ and $2/Q_0(0)$. $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. Then $N(2^2) \geq 1$ for $s \geq 5$.

If $4/A_4$, then $2/A_2$. By hypothesis I, there is a value of t for which $A_1 \equiv A_0 \equiv 1 \pmod{2}$, or $2/A_1$. If A_1 is odd, then $\theta = 0$, $Q_0(0) \equiv Q(0) \equiv 1 \pmod{2}$, and $N(2^2) \geq 1$ for $s \geq 5$. If A_1 is even, then $2/Q(1)$ or $2/Q(0)$ according as $2/A_0$ or $2/A_0$. Then $\theta = 1$ and, since $4/A_3$, either $2/Q_0(0)$ or $2/Q_0(1)$ according as $4/A_1$ or $4/A_1$. Then $N(2^3) \geq 1$ for $s \geq 9$.

- b. Suppose $2a_3 \equiv a_4 + 2a_2 \equiv 0 \pmod{4}$. Then $2/A_1$ for all values of t . If $4/a_4$, then $2/A_2$ and $\theta = 1$. $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. Since $4/A_3$, we have $2/Q_0(0)$ or $2/Q_0(1)$, according as $4/A_1$ or $4/A_1$ and $N(2^3) \geq 1$ for $s \geq 9$.

If $4/a_4$, then $2/A_2$. Choose $t \equiv 1 \pmod{2}$. Then $2/Q(0)$.

If $4/A_1$, then $\theta = 1$, $2/Q_0(0)$ and $N(2^3) \geq 1$ for $s \geq 9$. If $4/A_1$, $\theta \geq 2$.

If $8/A_1$, then $2/Q_0(0)$ and $N(2^4) \geq 1$ for $s \geq 17$.

If $8/A_1$, then $4/A_2$ since $a_4 \not\equiv 0 \pmod{8}$. Then $\theta = 2$, and since $8/A_3$, $2/Q_0(1)$. Then $N(2^4) \geq 1$ for $s \geq 17$.

3. Suppose $2/a_4$, then $v \equiv 4 \pmod{8}$ and $2/(A_4, A_3, A_2)$, $2/A_1$. Then $\theta = 0$ and $2/Q_0(0)$. $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. Then $N(2^2) \geq 1$ for $s \geq 5$.

Thus, with the three exceptions listed in the theorem $N(p^\gamma) \geq 1$, for polynomials of degree 4, $s \geq 21$, every prime p , and every integer n .

Then as in §5, with $\gamma_1 = 4$, $r_s(n) > 0$ for $n > C_2$.

8. Polynomials of degree 5. We prove

THEOREM 2. *If $P(y)$ satisfies hypotheses I and II, every sufficiently large integer can be represented as a sum of 53 values of a polynomial*

$$P(y) = a_5 \begin{bmatrix} y \\ 5 \end{bmatrix} + a_4 \begin{bmatrix} y \\ 4 \end{bmatrix} + a_3 \begin{bmatrix} y \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} y \\ 2 \end{bmatrix} + a_1 \begin{bmatrix} y \\ 1 \end{bmatrix},$$

the a 's being integers, $a_5 > 0$. $P(y)$ fails to satisfy hypothesis II when and only when one or more of the following eight sets of simultaneous conditions hold.

- 1) $9/a_5, 9/a_4, a_4 + 3a_2 \equiv a_3 + 6a_2 + 3a_1 \equiv 0 \pmod{9}$;
- 2) $9/(a_5, a_4), 3/(a_3, a_2), a_6 + 3a_3 \equiv 2a_5 + a_4 + 3a_3 + 3a_2$
 $\equiv 2a_3 + 3a_2 + 6a_1 \equiv 0 \pmod{27}$;
- 3) $2^3/(a_5, a_4), 2^4/a_5, a_3 \equiv a_2 + 2a_1 \equiv 0 \pmod{4}$;
- 4) $2^4/(a_5, a_4), 4/a_3, a_3 + 2a_2 \equiv a_2 + 2a_1 \equiv 0 \pmod{8}$;
- 5) $a_5 \equiv 2a_4 \equiv 16 \pmod{32}$,
 $2a_3 + 4a_2 \equiv a_4 + 2a_2 + 4a_1 \equiv 0 \pmod{16}$;

- 6) $2^5/a_5, a_4 \equiv 2a_3 \equiv 4a_2 \equiv 8 \pmod{16}, a_4 + a_3 + 6a_2 \equiv 0 \pmod{16},$
 $3a_4 + 4a_3 + 6a_2 + 12a_1 \equiv 2a_3 + 4a_2 \equiv 0 \pmod{32};$
- 7) 1. $a_5 \equiv 2, a_4 \equiv 0 \pmod{4}, a_3 \equiv 1, a_2 \equiv 0 \pmod{2},$
 $a_5 + a_4 + 6a_3 + 2a_2 + 4a_1 \equiv 0 \pmod{8},$
 or 2. $a_5 \equiv a_4 \equiv 2a_3 \equiv 2a_2 \equiv 2 \pmod{4}, 2/a_1,$
 $3a_4 + 4a_3 + 6a_2 \equiv 0 \pmod{8};$
- 8) $a_5 \equiv 4, a_4 \equiv 0 \pmod{8}, a_5 + a_4 + 6a_3 + 2a_2 + 12a_1 \equiv 0 \pmod{16}.$

First we prove that for $P(y)$ as in the theorem and $s \geq 53, N(p^\gamma) \geq 1.$

For $p > k$, this is proved in Lemma 4.

Let $p \leq k$. From formulas (5) and (6) we can calculate the coefficients $b_5, \dots, b_1$ and $A_5, \dots, A_0$. They are similar to those for $k = 4$ in §5, but for brevity, only the numerical coefficients in the expressions for them are indicated here. These are

$$b_5 = 1, \quad b_4 = 10, 5, \quad b_3 = 35, 30, 20, \quad b_2 = 50, 55, 60, 60,$$

$$b_1 = 24, 30, 40, 60, 120,$$

$$A_5 = 1, \quad A_4 = 5, 1, \quad A_3 = 10, 4, 1, \quad A_2 = 10, 6, 3, 1,$$

$$A_1 = 5, 4, 3, 2, 1, \quad A_0 = P(t).$$

A. Case $p = 5$.

1. Suppose $5/a_5$. Then $5/v$ and $t \equiv 0 \pmod{25}$ is admissible. Then $\theta \leq 1$ and since not all coefficients of $Q_0(x)$ are divisible by 5, for some value of x , $5/Q_0(x)$. Then $N(5^\gamma) \geq 1$ for $s \geq 26$.
2. Suppose $5/a_5$, then $5/v$ and $5/(A_5, A_4, A_3, A_2), 5/A_1$. Then $\theta = 0$, $5/Q_0(0)$, and $5/Q(0)$ or $5/Q(1)$ according as $5/A_0$ or $5/A_0$. Then $N(5) \geq 1$ for $s \geq 6$.

B. Case $p = 3$.

1. Suppose $3/(a_5, a_4, a_3)$. Then $3/v$ and every value of t is admissible.
 - a. Unless $b_5 = 4b_4 + 2b_2 \equiv b_1 \equiv 0 \pmod{9}$, let $t \equiv 0 \pmod{9}$. Then $\theta = 0$ and there is a value of x for which $3/Q_0(x)$. Then by Lemma 5, $N(3) \geq 1$ for $s \geq 4$.
 - b. Suppose $b_5 = 4b_4 + 2b_2 \equiv b_1 \equiv 0 \pmod{9}$. If $b_4 \not\equiv 0 \pmod{9}$, then $3/A_4$ and $\theta = 0$. Then for all values of t , $A_5 = 4A_4 + 2A_2 \equiv A_1 \equiv 0 \pmod{3}$, and the polynomial does not satisfy hypothesis II. This is exception 1 of the theorem.
 If $b_4 \equiv 0 \pmod{9}$, but $5b_5 + 3b_3 \equiv 4b_4 + 2b_2 \equiv b_1 \equiv 0 \pmod{27}$ does not hold, choose $t \equiv 0 \pmod{27}$. Then, since $3/b_3$, $\theta = 1$ and there is a value of x for which $3/Q_0(x)$. Then $N(3^2) \geq 1$ for $s \geq 10$. If $b_4 \equiv 0 \pmod{9}$ and $5b_5 + 3b_3 \equiv 4b_4 + 2b_2 \equiv b_1 \equiv 0 \pmod{27}$, then $\theta = 1$. Since, for all t , $5A_5 + 3A_3 = 4A_4 + 2A_2 = A_1 \equiv 0 \pmod{9}$, the poly-

nomial does not satisfy hypothesis II. This is exception 2 of the theorem.

2. Suppose $3/(a_5, a_4, a_3)$, then $v \equiv 3, 6 \pmod{9}$ and $3/(A_5, \dots, A_2)$. $3/A_1$ for every value of t implies $3/(a_5, a_4, a_3)$. Therefore t can be chosen so that $3/A_1$. Then $\theta = 0$, and $3/Q_0(0)$ and $3/Q(0)$ or $3/Q(1)$ according as $3/A_0$ or $3/A_0$. Then $N(3) \geq 1$ for $s \geq 4$.

C. Case $p = 2$.

1. Suppose $2^3/(a_5, a_4)$, $2/(a_3, a_2)$, then $2/v$ and every value of t is admissible.
 - a. Unless $a_3 \equiv a_2 + 2a_1 \equiv 0 \pmod{4}$, there is a value of t such that $2/A_1$. Then $\theta = 0$, $2/Q_0(0)$, and $N(2^2) \geq 1$ for $s \geq 5$.
 - b. If $a_3 \equiv a_2 + 2a_1 \equiv 0 \pmod{4}$ and $2^4/a_5$, the polynomial does not satisfy hypothesis II. This is exception 3 of the theorem.
 - c. Suppose $a_3 \equiv a_2 + 2a_1 \equiv 0 \pmod{4}$ and $2^4/a_5$.

If $2^4/a_4$, then $2/A_2$ and $\theta = 1$. If there is a value of t for which $4/A_1$, then $2/Q_0(0)$ and $N(2^3) \geq 1$ for $s \geq 9$. If $4/A_1$ for all t , then $2^3/a_3$, and $5A_5 + 3A_3 + 2A_2 \equiv 0 \pmod{4}$ for all t . The polynomial does not satisfy hypothesis II. This is exception 4 of the theorem.

If $2^4/a_4$, then $2/A_2$. If for some value of t , $4/A_1$, then $\theta = 1$ and $2/Q_0(0)$. Then $N(2^3) \geq 1$ for $s \geq 9$. If $4/A_1$ for all t , $2^3/a_3$. If $2^5/a_5$, $\theta = 1$ and $4/(5A_5 + 3A_3)$ for all t and the polynomial does not satisfy the hypothesis II. This is exception 5 of the theorem. If $2^5/a_5$, then $\theta = 2$. If for some value of t , $8/A_1$, $N(2^4) \geq 1$ for $s \geq 17$. If $8/A_1$ for all t , and $a_3 + 2a_2 \equiv 8 \pmod{16}$, then $8/(5A_5 + 4A_4 + 3A_3 + 2A_2)$. Then $N(2^4) \geq 1$ for $s \geq 17$. If $a_3 + 2a_2 \equiv 0 \pmod{16}$, the polynomial does not satisfy hypothesis II. This is exception 6 of the theorem.

2. Suppose $2/(a_5, a_4)$ but not all the conditions of C1 hold. Then $v \equiv 2 \pmod{4}$ and $2/(A_5, A_4, A_3)$.

- a. Unless $a_3 \equiv a_2 \equiv 0 \pmod{2}$, there is a value of t such that $2/(A_2 + A_1)$. If for the same value of t , $2/A_1$, then $\theta = 0$ and $2/Q_0(0)$. $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. Then $N(2^2) \geq 1$ for $s \geq 5$. If for every value of t for which $2/(A_2 + A_1)$, $2/A_1$, then $2/A_2$. Then $\theta = 1$ and $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. If $4/A_1$, we have $2/Q_0(0)$, and $N(2^3) \geq 1$ for $s \geq 9$. If $4/A_1$ and $4/a_5$, then $2/Q_0(1)$ and $N(2^3) \geq 1$ for $s \geq 9$. If $4/A_1$ and $4/a_5$, then $4/A_3$ and there is a value of t for which $2/A_2$. If for this value $2/A_0$, then $\theta = 1$ and $2/Q_0(1)$ or $2/Q_0(0)$ according as $4/A_1$ or $4/A_1$. $2/Q(0)$ and $N(2^2) \geq 1$ for $s \geq 5$. If for this value of t we have $2/A_0$, then the polynomial does not satisfy hypothesis II. This is exception 7 of the theorem.

- b. If $a_3 \equiv a_2 \equiv 0 \pmod{2}$, then $2/(A_2 + A_1)$. Choose $t \equiv 1 \pmod{2}$, then $2/Q(0)$.

If $a_5 \not\equiv a_4 \pmod{4}$, then $2/A_1$, $\theta = 0$, and $2/Q_0(0)$. Then $N(2^2) \geq 1$ for $s \geq 5$.

If $a_5 \equiv a_4 \pmod{4}$, $\theta \geq 1$. Then if $4/A_1$, $\theta = 1$ and $N(2^3) \geq 1$ for

$s \geq 9$. If $4/A_1$ and $4/a_5$, then $4/A_3$, $\theta = 1$ and $N(2^3) \geq 1$ for $s \geq 9$. If $4/A_1$ and $4/a_5$, $\theta = 2$. Then if $8/A_1$, $N(2^4) \geq 1$ for $s \geq 17$. If $8/A_1$ for all values of $t \equiv 1 \pmod{2}$, unless $8/a_5$ and $8/a_4$, $2/Q_0(1)$ and $N(2^4) \geq 1$ for $s \geq 17$. If $8/a_5$, $8/a_4$ the polynomial does not satisfy hypothesis II. This is exception 8 of the theorem.

3. Suppose $2/(a_5, a_4)$. Then $v \equiv 4 \pmod{8}$ and $2/(A_5, \dots, A_2)$. Since $2/A_1$ for all values of t implies $2/(a_5, a_4)$, there is a value of t for which $2/A_1$. Then $\theta = 0$, $2/Q_0(0)$ and $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. Then $N(2^2) \geq 1$ for $s \geq 5$.

Thus, with the eight exceptions listed in the theorem, $N(p^\gamma) \geq 1$, for polynomials of degree 5, $s \geq 53$, every prime p , and every integer n .

Then, as in §5, with $\gamma_1 = 5$, $r_s(n) > 0$ for $n > C_2$.

9. Polynomials of degree 6. We prove

THEOREM 3. *If $P(y)$ satisfies hypotheses I and II, every sufficiently large integer can be expressed as a sum of 133 values of a polynomial*

$$P(y) = a_6 \begin{bmatrix} y \\ 6 \end{bmatrix} + a_5 \begin{bmatrix} y \\ 5 \end{bmatrix} + a_4 \begin{bmatrix} y \\ 4 \end{bmatrix} + a_3 \begin{bmatrix} y \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} y \\ 2 \end{bmatrix} + a_1 \begin{bmatrix} y \\ 1 \end{bmatrix},$$

$a_6, \dots, a_1$ being integers, $a_6 > 0$.

It is assumed from the beginning that $P(y)$ satisfies hypothesis II. We prove first that for a polynomial as in the theorem and $s \geq 133$, $N(p^\gamma) \geq 1$.

For $p > k$ this is proved in Lemma 4.

Let $p \leq k$. Formulas (5) and (6) give for $k = 6$

$$b_6 = 1, \quad b_5 = 15, 6, \quad b_4 = 85, 60, 30, \quad b_3 = 225, 210, 180, 120,$$

$$b_2 = 274, 300, 330, 360, 360, \quad b_1 = 120, 144, 180, 240, 360, 720,$$

$$A_6 = 1, \quad A_5 = 6, 1, \quad A_4 = 15, 5, 1, \quad A_3 = 20, 10, 4, 1,$$

$$A_2 = 15, 10, 6, 3, 1, \quad A_1 = 6, 5, 4, 3, 2, 1, \quad A_0 = P(t).$$

A. Case $p = 5$.

1. Suppose $5/(a_6, a_5)$, then $5/v$ and every value of t is admissible. By hypothesis I, at least one of $A_6, \dots, A_1$ is not divisible by 5. Then $\theta \leq 1$. Choose t such that for some value of x , $5/Q_0(x)$. Then $N(5^\gamma) \geq 1$ for $s \geq 26$.
2. Suppose $5/(a_6, a_5)$, then $5/v$ and $5/(A_6, \dots, A_2)$. There is a value of t such that $5/A_1$, since $5/A_1$ for all t implies $5/(a_5, a_6)$. Therefore $\theta = 0$, $5/Q_0(0)$, and $5/Q(0)$ or $5/Q(1)$ according as $5/A_0$ or $5/A_0$. Then $N(5) \geq 1$ for $s \geq 6$.

B. Case $p = 3$.

1. Suppose $3^2/a_5$, $3/(a_5, a_4, a_3)$, then $3/v$. Then as in A1 above, $\theta \leq 1$ and $N(3^\gamma) \geq 1$ for $s \geq 10$.
2. Suppose not all conditions of B1 hold. Then $v \equiv 3, 6 \pmod{9}$ and $3/(A_6, \dots, A_3)$.

- a. If $3/a_6$, then $3/A_2$. If there is a value of t for which $3/A_1$, $\theta = 0$ and $3/Q_0(0)$. $3/Q(0)$ or $3/Q(1)$ according as $3/A_0$ or $3/A_0$. Then $N(3) \geq 1$ for $s \geq 4$. If $3/A_1$ for all t , then $3/(a_5, a_4, a_3)$. Then since $3^2/a_6$, $3^2/A_2$, and $\theta = 1$. Choose t so that $3/A_0$. Then $3/Q(0)$ and $3/Q_0(0)$ or $3/Q_0(1)$ according as $3^2/A_1$ or $3^2/A_1$. Then $N(3^2) \geq 1$ for $s \geq 10$.
- b. If $3/a_6$, then $3/A_2$ and $\theta = 0$. Then there is a value of x for which $3/Q(x)$. $3/Q_0(0)$ or $3/Q_0(1)$ according as $3/A_1$, or $3/A_1$. Then $N(3) \geq 1$ for $s \geq 4$.
- C. Case $p = 2$.
1. Suppose $2^4/a_6$, $2^3/(a_5, a_4)$, $2/(a_3, a_2)$. Then $2/v$. As in A1 $\theta \leq 2$ and $N(2^v) \geq 1$ for $s \geq 17$.
 2. Suppose $2/(a_6, a_5, a_4)$ and not all the conditions of C1 hold. Then $v \equiv 2 \pmod{4}$ and $2/A_6, \dots, A_4$.
 - a. If $2^2/a_6$, then $2/A_3$ and $\theta = 0$. Choose t admissible and such that for some value of x , $2/Q_0(x)$. Then $N(2^2) \geq 1$ for $s \geq 5$.
 - b. If $2^2/a_6$, $a_3 \not\equiv a_2$, $a_3 \equiv 0 \pmod{2}$, then for every value of t , $2/(A_2 + A_1)$. Choose t such that for some value of x , $2/Q_0(x)$. $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. Then $\theta \leq 1$ and $N(2^v) \geq 1$ for $s \geq 9$. If $a_3 \equiv 1 \pmod{2}$, choose $t \equiv 1 \pmod{2}$. Then $2/(A_2 + A_1)$. Since $t \equiv 0 \pmod{2}$ is not admissible, there is a value of x for which $2/Q_0(x)$. As above, $N(2^v) \geq 1$ for $s \geq 9$.
 - c. Let $2^2/a_6$, $a_3 \equiv a_2$, $a_3 \equiv 1 \pmod{2}$. If for $t \equiv 0 \pmod{2}$ there is a value of x for which $2/Q_0(x)$, then $\theta \leq 1$ and $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. Then $N(2^v) \geq 1$ for $s \geq 9$. Otherwise choose $t \equiv 1 \pmod{2}$. Since, if $a_3 \equiv 0 \pmod{2}$, $t \equiv 1 \pmod{2}$ is the only admissible value of t , we have now to consider the case $2^2/a_5$ and $a_3 \equiv a_2 \pmod{2}$, $t \equiv 1 \pmod{2}$. If $a_5 \not\equiv a_4 \pmod{4}$, then $2/A_1$. $\theta = 0$, $2/Q_0(0)$, $2/Q(0)$. Then $N(2^v) \geq 1$ for $s \geq 5$. If $a_5 \equiv a_4 \pmod{4}$ and $a_5 \not\equiv 0 \pmod{4}$, then $\theta \leq 5$, since for $t \equiv 1 \pmod{2}$, $2^6/(2A_6, A_5)$. Then $N(2^v) \geq 1$ for $s \geq 129$. If $a_5 \equiv a_4 \pmod{4}$ and $a_5 \equiv 0 \pmod{4}$, $\theta \leq 5$. For if $\theta = 6$, for $t \equiv 1 \pmod{2}$,

$$2^6/6A_6 \text{ implies } a_6 \equiv 0 \pmod{2^3},$$

$$2^6/5A_5 \text{ implies } 5a_6 + 6a_5 \equiv 0 \pmod{2^5},$$

$$2^6/4A_4 \text{ implies } 3a_6 + 2a_5 + 2a_4 \equiv 0 \pmod{2^4}.$$
 These together imply one of
 - 1) $16/a_6$, $8/(a_5, a_4)$. In this case the polynomial belongs in case C1.
 - 2) $16/a_6$, $8/a_5$, $8/a_4$. Then if $2/a_3$, $8/2A_2$; and if $2/a_3$, $8/A_3$. Thus $\theta \leq 5$, and $N(2^v) \geq 1$ for $s \geq 129$.
 3. Suppose $2/(a_6, a_5, a_4)$, then $v \equiv 4 \pmod{8}$. Then $2/(A_6, \dots, A_2)$. Unless $a_6 \equiv a_5 \equiv 1$, $a_4 \equiv 0 \pmod{2}$, there is a value of t for which $2/A_1$. Then $\theta = 0$, $2/Q_0(0)$, and $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. Then $N(2^2) \geq 1$ for $s \geq 5$.

If $a_6 \equiv a_5 \equiv 1$, $a_4 \equiv 0 \pmod{2}$, since $8 \nmid A_3$, $\theta \leq 2$. Choose t so that $2 \nmid A_0$ and for some value of x , $2 \nmid Q_0(x)$. Then $N(2^r) \geq 1$ for $s \geq 17$.

Thus, for $s \geq 133$, $N(p^r) \geq 1$, for polynomials of degree 6 satisfying hypotheses I and II.

Then as in §5, with $\gamma_1 = 5$, $r_s(n) > 0$ for $n > C_2$.

10. Polynomials of degree 7. We prove

THEOREM 4. *If $P(y)$ satisfies hypotheses I and II, every sufficiently large integer can be expressed as a sum of 325 values of a polynomial*

$$P(y) = a_7 \begin{bmatrix} y \\ 7 \end{bmatrix} + \cdots + a_1 \begin{bmatrix} y \\ 1 \end{bmatrix},$$

$a_7, \dots, a_1$ being integers, $a_7 > 0$.

It is assumed from the beginning that $P(y)$ satisfies hypothesis II. We prove first that for a polynomial as in the theorem and $s \geq 325$, $N(p^r) \geq 1$.

For $p > k$ this is proved in Lemma 4.

Let $p \leq k$. Formulas (5) and (6) give for $k = 7$

$$b_7 = 1, \quad b_6 = 21, 7, \quad b_5 = 175, 105, 42,$$

$$b_4 = 735, 595, 420, 210, \quad b_3 = 1624, 1575, 1470, 1260, 840,$$

$$b_2 = 1764, 1918, 2100, 2310, 2520, 2520,$$

$$b_1 = 720, 840, 1008, 1260, 1680, 2520, 5040,$$

$$A_7 = 1, \quad A_6 = 7, 1, \quad A_5 = 21, 6, 1, \quad A_4 = 35, 15, 5, 1,$$

$$A_3 = 35, 20, 10, 4, 1, \quad A_2 = 21, 15, 10, 6, 3, 1,$$

$$A_1 = 7, 6, 5, 4, 3, 2, 1, \quad A_0 = P(t).$$

A. Case $p = 7$.

1. Suppose $7 \nmid a_i$, then $7 \nmid v$ and every value of t is admissible. At least one of $A_7, \dots, A_1$ is not divisible by 7. Then $\theta \leq 1$ and $N(7^r) \geq 1$ for $s \geq 50$.
2. Suppose $7 \nmid a_7$. Then $v \equiv 7, 14, \dots, 42 \pmod{49}$, and $7 \nmid (A_7, \dots, A_2)$ but $7 \nmid A_1$. Thus $\theta = 0$, $7 \nmid Q_0(0)$ and $7 \nmid Q(0)$ or $7 \nmid Q(1)$ according as $7 \nmid A_0$ or $7 \mid A_0$. Then $N(7) \geq 1$ for $s \geq 8$.

B. Case $p = 5$.

1. Suppose $5 \nmid (a_7, a_6, a_5)$, then $5 \nmid v$. As in A1, $\theta \leq 1$. Then $N(5^r) \geq 1$ for $s \geq 26$.
2. Suppose $5 \nmid (a_7, a_6, a_5)$. Then $v \equiv 5, 10, 15, 20 \pmod{25}$, $5 \nmid (A_7, \dots, A_2)$ and there is a value of t for which $5 \nmid A_1$. Thus $\theta = 0$, $5 \nmid Q_0(0)$, and $5 \nmid Q(0)$ or $5 \nmid Q(1)$ according as $5 \nmid A_0$ or $5 \mid A_0$. Then $N(5) \geq 1$ for $s \geq 6$.

C. Case $p = 3$.

1. Suppose $3^2 \nmid (a_7, a_6)$, $3 \nmid (a_5, a_4, a_3)$, then $3 \nmid v$. As in A1 $\theta \leq 1$, and $N(3^r) \geq 1$ for $s \geq 10$.

2. Suppose not all conditions of case C1 hold. Then $v \equiv 3, 6 \pmod{9}$ and $3/(A_7, \dots, A_3)$. Unless $3/(a_7, a_6)$, there is a value of t for which $3/A_2$. Then $\theta = 0$ and there is a value of x for which $3/Q(x)$. $3/Q_0(0)$ or $3/Q_0(1)$ according as $3/A_1$ or $3/A_1$. Then $N(3) \geq 1$ for $s \geq 4$.

If $3/(a_7, a_6)$, then $3/A_2$ for all t . Unless $3/(a_5, a_4, a_3)$, there is a value of t for which $3/A_1$. Then $\theta = 0$, $3/Q_0(0)$, and there is a value of x for which $3/Q(x)$. Then $N(3) \geq 1$ for $s \geq 4$.

If $3/(a_7, a_5, a_4, a_3)$, then $3^2/(a_7, a_6)$. Choose t so that $3/Q(0)$, and that there is a value of x for which $3/Q_0(x)$. If $9/a_7$, then $9/a_6$ and $3^2/A_2$. Then $\theta \leq 2$ and $N(3^\gamma) \geq 1$ for $s \geq 10$. If $9/a_7$, then $\theta \leq 4$ since $3^5/A_5$, and $N(3^\gamma) \geq 1$ for $s \geq 244$.

D. Case $p = 2$.

1. Suppose $2^4/(a_7, a_6)$, $2^3/(a_5, a_4)$, $2/(a_3, a_2)$, then $2/v$. As in A1 $\theta \leq 2$ and $N(2^\gamma) \geq 1$ for $s \geq 17$.

2. Suppose $2/(a_7, \dots, a_4)$, and not all the conditions of case D1 hold, then $v \equiv 2 \pmod{4}$, and $2/(A_7, \dots, A_4)$. If $2/(a_3, a_2)$, there is at least one value of t for which $A_3 + A_2 + A_1 \not\equiv 0 \pmod{2}$. Then $2/Q(0)$ or $2/Q(1)$ according as $2/A_0$ or $2/A_0$. If for such a value of t there is a value of x for which $2/Q_0(x)$, since $2/(A_3, A_2, A_1)$, $\theta \geq 1$. Then $N(2^\gamma) \geq 1$ for $s \geq 9$. Otherwise, choose t such that $A_3 + A_2 + A_1 \equiv 0 \pmod{2}$, $2/A_0$ and there is a value of x for which $2/Q_0(x)$. In this case $2/a_3$. Then if $2/a_2$, the only value of t for which $A_3 + A_2 + A_1 \equiv 0 \pmod{2}$ is $t \equiv 0 \pmod{2}$ and this is not admissible. Thus $2/a_2$. We have left to consider then $a_3 \equiv a_2 \pmod{2}$ with $t \equiv 1 \pmod{2}$, the only admissible value of t . Then $2/Q(0)$ and there is a value of x for which $2/Q_0(x)$. $\theta \leq 6$, for if $\theta = 7$, $A_7 \equiv 2A_6 \equiv A_5 \equiv 4A_4 \equiv 0 \pmod{2^7}$, and these congruences imply $2^4/(a_7, a_6)$, $2^3/(a_5, a_4)$. If $a_3 \equiv a_2 \equiv 1 \pmod{2}$, $2^3/A_3$, and if $a_3 \equiv a_2 \equiv 0 \pmod{2}$, the polynomial belongs in case D1. Thus $\theta \leq 6$ and $N(2^\gamma) \geq 1$ for $s \geq 257$.

3. Suppose $2/(a_7, \dots, a_4)$, then $v \equiv 4 \pmod{8}$ and $2/(A_7, \dots, A_2)$. Unless $a_7 + a_6 + a_5 \equiv a_4 \equiv 0 \pmod{2}$, there is a value of t for which $2/A_1$. Then $\theta = 0$, $2/Q_0(0)$ and $2/Q(0)$ or $2/Q(1)$. Then $N(2^\gamma) \geq 1$ for $s \geq 5$. If $a_7 + a_6 + a_5 \equiv a_4 \equiv 0 \pmod{2}$, one of a_7, a_6, a_5 is even and the other two odd. Choose t so that $2/A_0$ and so that there is a value of x for which $2/Q_0(x)$. If $2/a_7$, then $8/A_3$. Then $\theta \leq 2$ and $N(2^\gamma) \geq 1$ for $s \geq 17$. If $2/a_6$ and $t \equiv 1 \pmod{2}$, then $8/A_3$. If $t \equiv 0 \pmod{2}$, $2^7/A_5$. In either case $\theta \leq 6$ and therefore $N(2^\gamma) \geq 1$ for $s \geq 257$. Similarly, if $2/a_5$, $\theta \leq 6$ and $N(2^\gamma) \geq 1$ for $s \geq 257$.

Thus, for $s \geq 325$, $N(p^\gamma) \geq 1$, for polynomials of degree 7 satisfying hypotheses I and II.

Then as in §5, with $\gamma_1 = 9$, $r_s(n) > 0$ for $n > C_2$.

11. Polynomials of degree k , $7 < k \leq 28$. We prove

THEOREM 5. If $P(y)$ satisfies hypotheses I and II and if for each prime $p \leq k$ one of the following conditions holds

(1) For some admissible value of t with which $P(y)$ satisfies hypothesis II, $p_f^t(A_k, \dots, A_1)$,

(2) For some admissible value of t , with which $P(y)$ satisfies hypothesis II, $\gamma \leq \gamma_0$, where γ_0 is the greatest value of γ such that $p^\gamma \leq (k-2)2^{k-1} + 5$, then every sufficiently large integer can be expressed as a sum of $s \geq (k-2)2^{k-1} + 5$ values of a polynomial

$$P(y) = a_k \begin{bmatrix} y \\ k \end{bmatrix} + \dots + a_1 \begin{bmatrix} y \\ 1 \end{bmatrix}, \quad 7 < k \leq 28,$$

$a_k, \dots, a_1$ being integers, $a_k > 0$.

We prove first that $N(p^\gamma) \geq 1$.

For $p > k$ this is proved by Lemma 4.

Let $p \leq k$. If condition (1) holds, $N(p^\gamma) \geq 1$ by Lemma 6. If condition (2) holds, $N(p^\gamma) \geq 1$ by Lemma 5.

Then as in §5, $r_s(n) > 0$ for $n > C_2$.

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VITA

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